Improvement of a Problem from American Mathematical Monthly

Abstract

In this note we give an improvement to a problem that was published in the American Mathematical Monthly.

1 Introduction

For a triangle ABC let A, B, C denote its angles, a, b, c the lengths of the corresponding sides, R and r the circumradius and inradius, respectively, and s the semiperimeter.
In triangle ABC the following inequality holds (see [1], [2], or [3]):

$$(1-\cos A)(1-\cos B)(1-\cos C) \geq \cos A \cos B \cos C. \tag{1.1}$$

In 2008, Cezar and Tudorel Lupu proposed the following problem (see [4])
For an acute triangle with side-lengths a, b, c, inradius r and semiperimeter s, prove that

$$(1-\cos A)(1-\cos B)(1-\cos C) \geq \cos A \cos B \cos C \left(2 - \frac{3\sqrt{3}r}{s}\right). \tag{1.2}$$

A solution based on Popoviciu's inequality was published in the October 2009 issue of the American Mathematical Monthly.
By inequality ([5]): $s \geq 3\sqrt{3}r$, we know that (1.2) is stronger than (1.1).
In this note, we give an improvement of (1.2).

2 Main results

Theorem 2.1. *In triangle ABC,*

$$(1-\cos A)(1-\cos B)(1-\cos C) \geq \cos A \cos B \cos C \left(2 - \frac{2r}{R}\right). \tag{2.1}$$

In order to prove Theorem 2.1, we need the following Lemma.

Lemma 2.1. *(See [5]) In triangle ABC, the following inequality holds.*

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}. \tag{2.2}$$

Proof of Theorem 2.1:

Proof. Because $1 - \cos x = 2\sin^2 \frac{x}{2}$, we have

$$(1-\cos A)(1-\cos B)(1-\cos C) = 8\sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2},$$

By the known identities in a triangle

$$\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{4R},$$
$$\cos A \cos B \cos C = \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2},$$

(2.1) is equivalent to

$$\frac{r^2}{2R^2} \geq \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2} \left(2 - \frac{2r}{R}\right)$$

$$\Leftrightarrow s^2 \leq 4R^2 + 4Rr + r^2 + \frac{Rr^2}{R-r}. \tag{2.3}$$

Using Lemma 2.1, it suffices to prove

$$2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)} \leq 4R^2 + 4Rr + r^2 + \frac{Rr^2}{R-r} \tag{2.4}$$

$$\Leftrightarrow 2(R-2r)\sqrt{R(R-2r)} \leq 2R^2 - 6Rr + 2r^2 + \frac{Rr^2}{R-r} \tag{2.5}$$

Because

$$\begin{aligned} &2R^2 - 6Rr + 2r^2 + \frac{Rr^2}{R-r} - 2(R-2r)\sqrt{R(R-2r)}, \\ &= 2(R-2r)(R-r) - \frac{R-2r)r^2}{R-r} - 2(R-2r)\sqrt{R(R-2r),} \\ &= \frac{R-2r}{R-r}\left[2(R-r)^2 - r^2 - 2(R-r)\sqrt{R(R-2r)}\right], \end{aligned}$$

we have

$$\begin{aligned} 2(R-r)^2 - r^2 &= 2R(R-2r) + r^2 > 0, \\ \left(2(R-r)^2 - r^2\right)^2 - \left(2(R-r)\sqrt{R(R-2r)}\right)^2 &= r^4 > 0, \end{aligned}$$

and by Euler's inequality $R \geq 2r$, we obtain

$$2R^2 - 6Rr + 2r^2 + \frac{Rr^2}{R-r} - 2(R-2r)\sqrt{R(R-2r)} \geq 0.$$

This completes the proof of (2.1). □

Remark 2.1. *If triangle ABC is acute, by inequality ([5]): $s \leq \frac{3\sqrt{3}R}{2}$, we conclude that (2.1) is stronger than (1.2)*

References

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